Homework 1
Real Analysis

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Note on notation: When I use the symbol $\subset$, it does not imply that the subset is proper. In writing $A \subset X$, I mean only that $a \in A \implies a \in X$, leaving open the possibility that $A = X$. I do not use the symbol $\subseteq$.

Proposition 0.1 (Exercise 1)**.** *The middle-thirds Cantor set $\mathcal{C}$ is totally disconnected and perfect.*

Proof. First we show that $\mathcal{C}$ is totally disconnected. Let $x, y \in \mathcal{C}$ such that $x \neq y$. Then $|x - y| > 0$, so there exists $k \in \mathbb{N}$ such that $(1/3)^k < |x - y|$ (because $\lim_k (1/3)^k = 0$). Since $(1/3)^k < |x - y|$, x and y must be in different subintervals of C_k, since the length of each interval in C_k is $(1/3)^k$. Thus there is a whole interval of points in $[0, 1] \setminus \mathcal{C}$ between x and y, so $\mathcal{C}$ is totally disconnected.

Now we show that $\mathcal{C}$ is perfect. Let $x \in \mathcal{C}$. We will show that x is not an isolated point by finding a nearby point in $\mathcal{C}$ in a ball of any radius centered at x. Let $r > 0$, then choose k large enough that $(1/3)^k \leq r$. Since $\mathcal{C} = \cap_j C_j$, we have $x \in C_k$. Since each subinterval of C_k has length $(1/3)^k$ and x is in one such subinterval. Let y be an endpoint of that subinterval so that $x \neq y$ (there are two endpoints, so this is always possible). Then we have $x \neq y$ and $|x - y| < (1/3)^k < r$, so $y \in B_r(x) \cap \mathcal{C}$. Thus x is not an isolated point, and hence $\mathcal{C}$ is perfect. $\square$

Proposition 0.2 (Exercise 2a)**.** *Let $\mathcal{C}$ be the middle-thirds Cantor set. Then for $x \in [0, 1]$, $x \in \mathcal{C}$ if and only if x has a ternary expansion*

$$x = \sum_{k=1}^{\infty} a_k 3^{-k}$$

where $a_k \in \{0, 2\}$ for all k.

Proof. First, suppose that $x \in [0, 1]$ has such a ternary expansion, where each $a_k \in \{0, 2\}$. Since $a_1 \in \{0, 2\}$, we know that $0 \leq x \leq 1/3$ or $2/3 \leq x \leq 1$, so $x \in C_1$. More generally, we can see by induction that $a_k \in \{0, 2\} \implies x \in C_k$. Thus $x \in C_k$ for all k, so $x \in \mathcal{C}$.

Now suppose that $x \in \mathcal{C}$. We know x has some ternary expansion $\sum_{k=1}^{\infty} a_k 3^{-k}$ with $a_k \in \{0, 1, 2\}$ for all k. Consider the remaining intervals after just the first stage of construction, $[0, 1/3] \cup [2/3, 1]$. If $x \in [0, 1/3)$, then $a_1 = 0$. If $x = 1/3$, then x also has the ternary expansion $\sum_{k=2}^{\infty} 2/3$. If $x \in [2/3, 1)$, then $a_1 = 2$, and if $x = 1$, then x has the ternary expansion $\sum_{k=1}^{\infty} 2/3$.

By induction on the recursive construction, at the kth stage, if x is in the left half of a "split" interval, then $a_k = 0$, except possibly at the right endpoint. But at that endpoint, x has a finite ternary expansion terminating with a $1/3^k$ term. But this term can be rewritten as a ternary expansion involving only 0's and 2's by the following substitution:

$$\frac{1}{3^k} = \sum_{j=k+1}^{\infty} \frac{2}{3^j}$$

And for x in the right half of a "split" interval, we get $a_k = 2$, except possibly at the right endpoint. But again, if we get a ternary expansion involving a 1 at such an endpoint, it must be a finite expansion terminating in a $1/3^k$ term, which we can expand as above. Thus x has the required ternary expansion. □

Proposition 0.3 (Exercise 2b)**.** *Let F be the Cantor-Lebesgue function on $\mathcal{C}$. F is well defined and continuous, and $F(0) = 0$, and $F(1) = 1$.*

Proof. First we show that F is well defined. To do this, we need to show that for $x = \sum_{k=1}^{\infty} a_k 3^{-k} \in \mathcal{C}$, the value of F converges. By the previous proposition, we know that $x \in \mathcal{C}$ has a ternary expansion where each a_k is 0 or 2, so if we define $b_k = a_k/2$, each b_k is 0 or 1. Then $b_k/2^k \leq 1/2^k$ for each k. We know that the geometric series $\sum_{k=1}^{\infty} 1/2^k$ converges, so by the comparison test,

$$F(x) = \sum_{k=1}^{\infty} \frac{b_k}{2^k}$$

converges as well. Thus F is well defined.

Now we show that F is continuous on $\mathcal{C}$. Let $x_0 \in \mathcal{C}$. To show that F is continuous at x_0, we will show that for $\epsilon > 0$, there exists $\delta > 0$ such that

$$|x - x_0| < \delta \implies |F(x) - F(x_0)| < \epsilon$$

Fix $\epsilon > 0$. Then there exists $n \in \mathbb{N}$ such that $2^{-n-1} < \epsilon < 2^{-n}$. Let $\delta = 3^{-n}$. Let

$$x = \sum_{k=1}^{\infty} a_k 3^{-k} \qquad x_0 = \sum_{k=1}^{\infty} c_k 3^{-k}$$

For convenience, let $b_k = a_k - c_k$. Then

$$|x - x_0| = \left| \sum_{k=1}^{\infty} a_k 3^{-k} - \sum_{k=1}^{\infty} c_k 3^{-k} \right| = \left| \sum_{k=1}^{\infty} (a_k - c_k) 3^{-k} \right| = \left| \sum_{k=1}^{\infty} b_k 3^{-k} \right|$$

If $|x - x_0| < \delta = 3^{-n}$, then $b_1, b_2, \ldots b_n$ are all zero. Then the first n terms of $|F(x) - F(x_0)|$ are also zero:

$$|F(x) - F(x_0)| = \left| \sum_{k=1}^{\infty} (a_k/2) 2^{-k} - \sum_{k=1}^{\infty} (c_k/2) 2^{-k} \right| = \left| \sum_{k=1}^{\infty} (a_k - c_k) 2^{-k-1} \right|$$

$$= \left| \sum_{k=1}^{\infty} b_k 2^{-k-1} \right| = \left| \sum_{k=n+1}^{\infty} b_k 2^{-k-1} \right| \leq 2^{-n-1} < \epsilon$$

Thus F is continuous at x_0.

Now we show that $F(0) = 0$ and $F(1) = 1$. In the ternary expansion for zero, each $a_k = 0$. In the ternary expansion of 1, each $a_k = 2$. Then

$$0 = \sum_{k=1}^{\infty} (0) 3^{-k} \quad \rightsquigarrow \quad F(0) = \sum_{k=1}^{\infty} (0) 2^{-k} = \sum_{k=1}^{\infty} 0 = 0$$
$$1 = \sum_{k=1}^{\infty} (2) 3^{-k} \quad \rightsquigarrow \quad F(1) = \sum_{k=1}^{\infty} (1) 2^{-k} = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k = 1$$

since both are geometric series. □

Proposition 0.4 (Exercise 2c)**.** *The Cantor-Lebesgue function $F : \mathcal{C} \to [0,1]$ is surjective.*

Proof. Let $y \in [0,1]$. Then y has a binary expansion $y = \sum_{k=1}^{\infty} b_k 2^{-k}$, where each b_k is zero or one. Let $a_k = 2b_k$ for each k. Then let $x = \sum_{k=1}^{\infty} a_k 3^{-k}$. We can see from the definition of F that $F(x) = y$. Furthermore, each a_k is zero or two, so x is in the Cantor set by Exercise 2a. Hence F is surjective. □

Proposition 0.5 (Exercise 3a)**.** *The complement of $\mathcal{C}_\xi$ in $[0,1]$ is the union of open intervals of total length 1.*

Proof. At each stage of the construction, we remove an open interval of length ξ/l from the middle of each remaining interval, where l is the length of each interval. This means that at each stage, the total length of each interval after the removal is $l(1-\xi)$. Since each closed interval from which we remove this interval is the same, the total sum of the remaining intervals after each stage is multiplied by $(1-\xi)$. We begin with $[0,1]$, so after the first stage, the remaining length is $(1-\xi)$, and after the second stage the remaining length is $(1-\xi)^2$. Generally, the remaining length after the kth stage is $(1-\xi)^k$. Thus, the sum of the lengths of all removed intervals up to the kth stage is $1-(1-\xi)^k$. In the limit as $k \to \infty$, this length of remaining intervals goes to 1 for $\xi \in (0,1)$. □

Proposition 0.6 (Exercise 3b)**.** $m_*(\mathcal{C}_\xi) = 0$

Proof. After k iterations of the removal process, there are 2^k intervals each with length

$$\frac{1-(1-\xi)^k}{2^k}$$

and the total length of these intervals is $1-(1-\xi)^k$. That is, for each $k \in \mathbb{N}$, there is a covering of $\mathcal{C}_\xi$ of disjoint cubes (intervals) Q_j with total length $1-(1-\xi)^k$. Thus $\inf \left\{ \sum_{j=1}^{\infty} |Q_j| \right\}$ is less than or equal to $1-(1-\xi)^k$ for all $k \in \mathbb{N}$. Since $\lim_k 1-(1-\xi)^k = 0$, this means that this infimum is at most zero. But the infimum cannot be less than zero, since it is an infimum over sums in which all terms are positive. Thus $m_*(\mathcal{C}_\xi) = 0$. □

Proposition 0.7 (Exercise 4a)**.** *Let $\hat{\mathcal{C}}$ be the Cantor-type set with constants $(l_k)_{k=1}^{\infty}$, such that $\sum_{k=1}^{\infty} l_k 2^{k-1} < 1$. Then*

$$m(\hat{\mathcal{C}}) = 1 - \sum_{k=1}^{\infty} l_k 2^{k-1}$$

Proof. The complement of $\hat{\mathcal{C}}$ in $[0,1]$ is a countable union of disjoint intervals:

$$[0,1] \setminus \hat{\mathcal{C}} = B(c_{1,1}, l_1/2) \cup B(c_{2,1}, l_2/2) \cup B(c_{2,2}, l_2/3) \cup \ldots = \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{2^{k-1}} B(c_{k,j}, l_k/2)$$

Where $c_{k,j}$ is the center of the jth interval with length l_k. Each of these intervals is measurable, with measure l_k for some k. There are 2^{k-1} such intervals of length l_k for each k. Thus by countable additivity of m, we have

$$[0,1] \setminus \hat{\mathcal{C}} = \sum_{k=1}^{\infty} l_k 2^{k-1}$$

which converges to less than 1 by hypothesis. By additivity,

$$m([0,1] \setminus \hat{\mathcal{C}}) + m(\hat{\mathcal{C}}) = m([0,1]) = 1$$

Thus

$$m(\hat{\mathcal{C}}) = 1 - m([0,1] \setminus \hat{\mathcal{C}}) = 1 - \sum_{k=1}^{\infty} l_k 2^{k-1}$$

□

Proposition 0.8 (Exercise 4b)**.** *If $x \in \hat{\mathcal{C}}$, then there exists a sequence $(x_n)_{n=1}^{\infty}$ such that $x_n \notin \hat{\mathcal{C}}$ and $x_n \to x$ and $x_n \in I_n$ where I_n is a subinterval in the complement of $\hat{\mathcal{C}}$ with $|I_n| \to 0$.*

Proof. Let $x \in \hat{\mathcal{C}}$. At each stage of the construction of $\hat{\mathcal{C}}$, there remain 2^{-1} disjoint closed intervals, and x is in exactly one of these intervals. At the kth stage, that closed interval has length

$$\frac{1 - \sum_{j=1}^{k} l_k 2^{k-1}}{2^{k-1}}$$

since the numerator is the total length of remaining intervals, and the denominator is the number of remaining intervals. When we remove the open middle interval of length l_k from that closed interval, let x_k be the midpoint of that interval. Then

$$|x_k - x| \leq \frac{1 - \sum_{j=1}^{k} l_k 2^{k-1}}{2^{k-1}} \leq \frac{1}{2^{k-1}}$$

Thus $x_k \to x$, and by construction $x_k \notin \hat{\mathcal{C}}$. We let I_k be the interval of which x_k is the midpoint, so $I_k \subset (\hat{\mathcal{C}})^c$. We know that $|I_k| \to 0$ since $|I_k| = l_k$ and by definition of $\hat{\mathcal{C}}$, we have

$$\sum_{j=1}^{k} l_j 2^{j-1} < 1$$

for all k. □

Proposition 0.9 (Exercise 4c)**.** *$\hat{\mathcal{C}}$ is perfect, and contains no open interval.*

Proof. Let $x \in \hat{\mathcal{C}}$. By part (b), there exists a sequence $(x_n)_{n=1}^{\infty}$ with $x_n \notin \hat{\mathcal{C}}$ and $x_n \to x$. Thus, for every $r > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N \implies x_n \in B(x, r)$. Hence x is not an isolated point, so $\hat{\mathcal{C}}$ has no isolated points. It is clear that $\hat{\mathcal{C}}$ is closed, since its complement is a union of open intervals by construction. Thus $\hat{\mathcal{C}}$ is perfect.

Now we show that $\hat{\mathcal{C}}$ contains no open interval. Suppose that $\hat{\mathcal{C}}$ contains an open interval (a, b), and let $x \in (a, b)$. Then by part (b) we have a sequence (x_n) with $x_n \notin \hat{\mathcal{C}}$ and $x_n \to x$. Let $r = \min(|x - a|, |x - b|)$. Then there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $x_n \in B(x, r)$. But by construction of r, $B(x, r) \subset (a, b)$, so for $n \geq N$, $x_n \in (a, b)$. This is a contradiction, since $x_n \notin \hat{\mathcal{C}}$ and $(a, b) \subset \hat{\mathcal{C}}$. Thus $\hat{\mathcal{C}}$ contains no open interval. □

Proposition 0.10 (Exercise 4d)**.** *$\hat{\mathcal{C}}$ is uncountable.*

Proof. Let $x \in \hat{\mathcal{C}}$. At the kth stage of construction of $\hat{\mathcal{C}}$, we separate each closed subinterval into two subintervals. We define a funtion $f : \hat{\mathcal{C}} \to [0, 1]$ by

$$f(x) = \sum_{k=1}^{\infty} \frac{a_k}{2^k}$$

where $a_k = 0$ if x is in a left subinterval at stage k, and $a_k = 1$ if x is in a right subinterval at stage k. We claim that f is surjective. To see this, let $y \in [0, 1]$. Then y has a (not necessarily unique) binary expansion

$$y = \sum_{k=1}^{\infty} \frac{b_k}{2^k}$$

Then we use the b_k to choose a subinterval remaining at each stage of the construction of $\hat{\mathcal{C}}$. Let I_k be this chosen subinterval. For example, if $b_1 = 0$, we choose the left of the two subintervals left after the first removal of a middle interval of length l_1. Now we take the intersection over all such intervals, $\cap_{k=1}^{\infty} I_k$, and we know that this intersection is non-empty, since the endpoints of each I_k are never removed. If we choose some $x \in \cap_k I_k$, then $f(x) = y$ by construction. Hence f is surjective.

Since there is a surjective function $\hat{\mathcal{C}} \mapsto [0, 1]$, we know that $\hat{\mathcal{C}}$ cannot have smaller cardinality than $[0, 1]$. Hence $\hat{\mathcal{C}}$ is uncountable. □

Proposition 0.11 (Exercise 5a)**.** *Let $E \subset \mathbb{R}^d$ and let*

$$\mathcal{O}_n = \{x : d(x, E) < 1/n\}$$

If E is compact, then

$$m(E) = \lim_{n\to\infty} m(\mathcal{O}_n)$$

Proof. Let $E \subset \mathbb{R}^d$ be compact. We have $\mathcal{O}_n \supset \mathcal{O}_{n+1}$ and clearly $E \subset \cap_n \mathcal{O}_n$. We also claim that $\cap_n \mathcal{O}_n \subset E$. Suppose that $x \in \cap_n \mathcal{O}_n$. Then

$$d(x, E) = \inf\{d(x, y) : y \in E\} < \frac{1}{n}$$

for all $n \in \mathbb{N}$. Thus, for every n, the $B(x, 1/n) \cap E \neq \emptyset$. Thus x is a limit point of E. Since E is closed, it contains all of its limit points, so $x \in E$. Thus $\cap_n \mathcal{O}_n \subset E$, so we have equality.

Thus we have a sequence $\mathcal{O}_n$ of measurable sets with $\mathcal{O}_n \searrow E$. Since E is closed and bounded, each $\mathcal{O}_n$ is also bounded, so $m(\mathcal{O}_n) < \infty$ for all n. Then by Corollary 3.3, it follows that

$$m(E) = \lim_{n\to\infty} m(\mathcal{O}_n)$$

□

Proposition 0.12 (Exercise 5b). *The above proposition need not hold when E is closed and unbounded, or when E is open and bounded.*

Proof. First, we let E be the closed and unbounded set $\mathbb{Q}$ lying in $\mathbb{R}$. We know that $m(\mathbb{Q}) = 0$. (One way to see this is by using Theorem 3.2 on the collection of singleton sets of rationals.) However, $\mathcal{O}_n = \{x : d(x, \mathbb{Q}) < 1/n\}$ is $\mathbb{R}$ for all n, since no matter how small $1/n$, every real number is always within $1/n$ of some rational. But $m(\mathbb{R}) = \infty$, so $\lim_n m(\mathcal{O}_n) = \infty$.

Now we construct an open, bounded set that doesn't satisfy the above property. Let $\epsilon > 0$. Consider the rationals in $[0, 1]$, and order them $q_1, q_2, \ldots$. Around q_k, form an open ball of radius $2^{-k}\epsilon$. Then let

$$E = \bigcup_{k=1}^{\infty} B(q_k, 2^{-k}\epsilon)$$

E is open, since it is a union of open sets, and E is bounded within $[-1/2, 3/2]$. We have an upper bound on the measure of E by $m(E) \leq \sum_{k=1}^{\infty} 2^{-k+1}\epsilon = 2\epsilon$. Now consider $\mathcal{O}_n$. For each k, we can see that $B(q_k, 2^{-k}\epsilon + 1/n) \subset \mathcal{O}_n$, since everything in $B(q_k, 2^{-k}\epsilon + 1/n)$ is within $1/n$ of $B(q_k, 2^{-k}\epsilon)$. But for any $x \in [0, 1]$, there is a rational within $1/n$ of x, so

$$[0, 1] \subset \bigcup_{k=1}^{\infty} B(q_k, 2^{-k}\epsilon + 1/n)$$

Thus $m(\mathcal{O}_n) \geq 1$ for all n. Hence $\lim_n m(\mathcal{O}_n) \geq 1$. So for any $\epsilon < 1/2$, we have $m(E) \leq 1/2$ but $\lim_n m(\mathcal{O}_n) \geq 1$. □

Proposition 0.13 (Exercise 6). *Let $B_1 \subset \mathbb{R}^d$ be the unit ball, $B_1 = \{x \in \mathbb{R}^d : |x| < 1\}$. Let $B = B(x, r)$ be any ball in $\mathbb{R}^d$ with center x and radius $r > 0$. Then $m(B) = r^d m(B_1)$.*

Proof. We observe that $B = x + rB_1$, that is, B is equal to a translation of a dilation of the unit ball. By translation and dilation properties of m, we have

$$m(B) = m(x + rB_1) = m(rB_1) = r^d m(B_1)$$

□

Proposition 0.14 (Exercise 7)**.** *Let $E \subset \mathbb{R}^d$ be measurable, and let $\delta = (\delta_1, \delta_2, \dots \delta_d)$ with each $\delta_i > 0$. We define*

$$\delta E = \{(\delta_1 x_1, \delta_2 x_2, \dots \delta_d x_d) : (x_1, x_2, \dots x_d) \in E\}$$

Then δE is measurable, and more specifically,

$$m(\delta E) = \left(\prod_{i=1}^{d} \delta_i \right) m(E)$$

Proof. First we observe that if $Q = \prod_{i=1}^{d} [a_i, b_i]$ is a closed rectangle, then $\delta Q = \prod_{i=1}^{d} [\delta_i a_i, \delta_i b_i]$. Then

$$|\delta Q| = \prod_{i=1}^{d} (\delta_i b_i - \delta_i a_i) = \prod_{i=1}^{d} \delta_i (b_i - a_i) = \left(\prod_{i=1}^{d} \delta_i \right) \left(\prod_{i=1}^{d} (b_i - a_i) \right) = \left(\prod_{i=1}^{d} \delta_i \right) |Q|$$

Let $\epsilon > 0$. Since E is measurable, there exists an open set $\mathcal{O}$ such that $m_*(E \setminus \mathcal{O}) \leq \epsilon$. Let $\{Q_j\}_{j=1}^{\infty}$ be a covering of $E \setminus \mathcal{O}$ by closed cubes. Then $\{\delta Q_j\}$ is a covering of $\delta E \setminus \delta \mathcal{O}$ by closed cubes. (To see this, let $y \in \delta E \setminus \delta \mathcal{O}$. Then $y = (\delta_1 x_1, \dots \delta_d x_d)$ where $x = (x_1, \dots x_d) \in E \setminus \mathcal{O}$. Then $x \in Q_j$ for some j, so then $y \in \delta Q_j$.) We note that $\delta \mathcal{O}$ must be open since $\mathcal{O}$ is open. Thus

$$m_*(\delta E \setminus \delta \mathcal{O}) \leq \sum_{j=1}^{\infty} |\delta Q_j| = \left(\prod_{i=1}^{d} \delta_i \right) \sum_{j=1}^{\infty} |Q_j| \leq \left(\prod_{i=1}^{d} \delta_i \right) \epsilon$$

Since $\prod_i \delta_i$ is some constant, we can make $m_*(\delta E \setminus \delta \mathcal{O})$ arbitrarily small, hence δE is measurable.

Now let $\{Q_j\}_{j=1}^{\infty}$ be a covering of E by closed cubes. Then $\{\delta Q_j\}$ is a cover of δE by closed cubes. (For analogous reasons to the above parenthetical note.) We also define

$$\delta^{-1} E = \{(\delta_1^{-1} x_1, \dots \delta_d^{-1} x_d) : (x_1, \dots x_d) \in E\}$$

Using this notation, we can say that if $\{R_j\}_{j=1}^{\infty}$ is a covering of δE by closed cubes, then $\{\delta^{-1} R_j\}$ is a cover of E by closed cubes. With this identification, we can say that for any covering $\{Q_j\}$ of E by closed cubes,

$$\begin{aligned} m(\delta E) &= \inf \left\{ \sum_{j=1}^{\infty} |\delta Q_j| \right\} = \inf \left\{ \prod_{i=1}^{d} \delta_i \sum_{j=1}^{\infty} |Q_j| \right\} \\ &= \left(\prod_{i=1}^{d} \delta_i \right) \inf \left\{ \sum_{j=1}^{\infty} |Q_j| \right\} = \left(\prod_{i=1}^{d} \delta_i \right) m(E) \end{aligned}$$

□